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Lee-Yang edge singularity in the three-dimensional Gross-Neveu model at finite temperature

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Abstract

We discuss the relevance of the Lee-Yang edge singularity to the finite-temperature Z_2-symmetry restoration transition of the Gross-Neveu model in three dimensions. We present an explicit result for its large-N free-energy density in terms of $\zeta(3)$ and the absolute maximum of Clausen's function.

The Gross-Neveu model in $d = 3$ dimensions provides a remarkable example of a second order temperature driven phase transition in a theory which also exhibits dynamical symmetry breaking. The latter property is purely quantum field theoretical while the former one involves classical thermal fluctuations. We consider here the standard Lagrangian

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describing the Euclidean version of the Gross-Neveu model with $U(2N)$ symmetry [1–3]

$$\mathcal{L} = \bar{\psi}^a \partial\!\!\!/ \psi^a + \frac{G_0}{2N} (\bar{\psi}^a \psi^a)^2 , \qquad a = 1, 2, .., N , \tag{1}$$

where G_0 is the coupling. The partition function for the theory can be written, after integrating out the fundamental four-component massless Dirac fermions $\bar{\psi}^a$, ψ^a, with the help of the auxiliary scalar field $\sigma(x)$ as [1–3]

$$Z_\sigma[G_0] = \int (\mathcal{D}\sigma)\, e^{N\left[2\,\mathrm{Tr}[\ln(-\partial^2+\sigma^2)] - \frac{1}{2G_0}\int \mathrm{d}^3x\, \sigma^2(x)\right]} . \tag{2}$$

The model possesses a discrete Z_2 "chiral" symmetry as (1) is invariant under $\psi \to \gamma_5 \psi$. The usual $1/N$ expansion is generated if one expands as $\sigma(x) = \sigma_0 + O(1/\sqrt{N})$, provided that σ_0 satisfies the gap equation

$$\frac{\sigma_0}{G_0} = \int \frac{\mathrm{d}^3p}{(2\pi)^3} \frac{4\sigma_0}{p^2 + \sigma_0^2} . \tag{3}$$

One renormalizes (3) by introducing an UV cut-off Λ as

$$\frac{1}{G_0} = \frac{1}{G_*} + \frac{1}{G_R} = 4 \int^\Lambda \frac{\mathrm{d}^3p}{(2\pi)^3} \frac{1}{p^2} + \frac{1}{G_R} , \tag{4}$$

and obtains the renormalized coupling $1/G_R = -M/\pi$, where M is the arbitrary mass scale introduced by renormalization. From (2) and (4) it is easy to calculate the leading-N renormalized "effective action" $V_R(\sigma_0; G_R)$ defined as

$$\int \mathrm{d}^3x\, V_R(\sigma_0; G_R) = -\ln \left[\frac{Z_{\sigma_0}[G_R]}{Z_0[0]} \right] , \tag{5}$$

where the subtraction on the r.h.s. of (5) ensures a finite result. The result is [2]

$$V_R(\sigma_0, G_R) = \frac{N}{2\pi} \left(\frac{2}{3} |\sigma_0|^3 - M\sigma_0^2 \right) . \tag{6}$$

From (6) we can clearly separate three regimes: **A)** For $M < 0$ the minimum of (6) is always at the origin and the theory is in the Z_2-*symmetric phase* with $\sigma_0 = 0$. **B)** For $M > 0$ the minimum of (6) is at $\sigma_0 = M$, the theory is in the Z_2-*broken phase* and M can be identified as the mass of the elementary fermionic fields. **C)** Finally, for $M = 0$,

the theory is at the *critical point* and it is a non-trivial three-dimensional conformal field theory (CFT).

Notice that the UV subtraction prescription in (5) has generated the $|\sigma_0|^3$ term in (6). This term manifests itself as the dominant contribution at the critical point $M = 0$. It is then conceivable that, to leading-N, the critical behavior of the model is somehow related to a ϕ^3 theory. Of course, the true critical ground state is at $\sigma_0 = 0$ which would correspond to the zero-coupling critical point of the ϕ^3 theory, or equivalently to a free-field theory. This is consistent with the well known mean field theory behavior, to leading-N, of all the critical quantities of the model [2].

On the other hand, it is well known [4, 5] that the IR limit of the theory with action

$$S = -\int [\frac{1}{2}(\partial\phi)^2 + \mathrm{i}(h - h_c)\phi + \frac{1}{3!}\lambda\phi^3]\,\mathrm{d}^d x\,, \tag{7}$$

dictates the critical behavior of an Ising model in a purely imaginary magnetic field $\mathrm{i}h_c$ as the critical temperature is approached from above (from the symmetric phase). This critical point (Lee-Yang edge singularity [6]) corresponds to a non-unitary theory as it involves an imaginary coupling constant λ.

If the critical behavior of the Gross-Neveu model is in any way related to a ϕ^3 theory, one would expect that the Lee-Yang singularity might become relevant as one approaches the critical point of the model from a suitable symmetric phase. To investigate such a possibility we introduce the ingredient of temperature T by putting the model (2) in a slab geometry with one finite dimension of length $L = 1/T$. A crucial point is that the renormalization (4) is unaffected, since renormalizing the theory in the bulk suffices to remove the UV-divergences for finite temperature [1]. However, the gap equation (3) now becomes

$$\begin{aligned} \frac{\sigma_0}{G_0} &= 4\frac{\sigma_0}{L}\sum_{n=0}^{\infty}\int^{\Lambda}\frac{\mathrm{d}^2p}{(2\pi)^2}\frac{1}{p^2+\omega_n^2+\sigma_0^2} \\ &= \frac{\sigma_0}{G_*} - \frac{\sigma_0^2}{\pi} - \frac{2\sigma_0}{\pi L}\ln\left(1+e^{-L\sigma_0}\right)\,. \end{aligned} \tag{8}$$

Then, from (2) and (8) we can explicitly calculate the leading-N renormalized "effective

action” $V_R(\sigma_0, L; G_R)$ - now depending in addition on the “inverse” temperature L - as

$$
\begin{aligned}
V_R(\sigma_0, L; G_R) &= \frac{N}{2\pi L^3}\left[\frac{2}{3}\sigma_0^3 L^3 - M\sigma_0^2 L^3\right. \\
&\left. +4Li_3\left(-e^{-L\sigma_0}\right) - 4\ln\left(e^{-L\sigma_0}\right)Li_2\left(-e^{-L\sigma_0}\right)\right] \qquad (9)
\end{aligned}
$$

where $Li_n(z)$ are the standard polylogarithms [7]. This “effective action” presents a remarkably explicit example of high-temperature symmetry restoration in a *quantum* field theoretic system[3] which is *ordered* $(M > 0)$ at $T = 0$. The critical temperature is $1/L_c = T_c = M/2\ln 2$ [2].

When $M = 0$ in (9), then for all $T > 0$ we approach the critical point from the symmetric phase. This is the regime where we would expect the appearance of the Lee-Yang singularity. This seems rather difficult to imagine as, despite the appearance in (9) of the cubic term σ_0^3 as a result of the UV subtraction prescription (5), the coefficient of this term is real. Nevertheless, one can show that (9) for $M = 0$ is in fact an *even* function of σ_0. To see this we express (9) in terms of Nielsen’s generalized polylogarithms $S_{n,p}(z)$ [8] as follows

$$
V_R(\sigma_0, L; 0) = \frac{2N}{\pi L^3}\left[S_{1,2}(z) + S_{1,2}\left(\frac{1}{z}\right) - \zeta(3)\right] \qquad (10)
$$

$$
S_{1,2}(z) = \frac{1}{2}\int_0^z \frac{\ln^2(1-y)}{y}\,\mathrm{d}y \qquad (11)
$$

$$
S_{1,2}(1) = 8\,S_{1,2}(-1) = \zeta(3) \qquad (12)
$$

where we have set $z = -e^{-L\sigma_0}$. From (10) we see that $V_R(-\sigma_0, L; 0) = V_R(\sigma_0, L; 0)$. This remarkable property means that although the $L \to \infty$ $(T \to 0)$ behavior of (9) looks like its is dominated by the cubic term (with real coefficient and ground state at $\sigma_0 = 0$), fluctuations become important for *all* $T > 0$ and completely change the relevant underlying effective potential. To this end we point out that the step from (9) to (10) involves an all order resummation in σ_0. drive the theory towards another critical point.

[3] Notice that although we are dealing with symmetry restoration in two dimension, the Mermin-Wanger-Coleman theorem is not violated as the relevant symmetry is discrete (Z_2 here).

From (10). Then, from (10) we conclude that away from $\sigma_0 = 0$ the critical theory is described by an effective Hamiltonian which is an *even* function of σ_0. If we view now σ_0 as a *scalar order parameter* and couple it to an *external magnetic field*, the critical behavior of such a system in the high temperature phase can be shown to correspond to a ϕ^3 theory with purely imaginary coupling [4]. The critical point is determined by the non-zero solution of the gap equation (8) as

$$\sigma_0 \left[\sigma_0 + \frac{2}{L} \ln\left(1 + e^{-\sigma_0 L}\right) \right] = 0 \Rightarrow \sigma_0 = \pm \mathrm{i} \frac{2\pi}{3L} \,, \tag{13}$$

where we restricted $-\mathrm{i}\pi < L\sigma_0 < \mathrm{i}\pi$ to avoid the cut of the logarithm. The fact that σ_0 is now purely imaginary, however, does not affect the reality properties of the effective potential and we obtain

$$V_R(\pm \mathrm{i}\frac{2\pi}{3L}, L; 0) = \frac{N}{2\pi L^3} \left[\frac{4}{3}\zeta(3) - \frac{8\pi}{3} Cl_2\left(\frac{\pi}{3}\right) \right] \,, \tag{14}$$

where $Cl_2(\theta) = \mathrm{Im}\left[Li_2(e^{\mathrm{i}\theta})\right]$ is Clausen's function [7]. It is amusing to point out that $Cl_2(\pi/3) \approx 1.014942..$ is the absolute maximum of Clausen's function which is a well-documented numerical constant.

Our result (14) corresponds to the leading-N free-energy density of the Lee-Yang edge singularity in $d = 3$. The parameter N should not be confused with the number of components of the underlying order parameter [12], but should be regarded as a suitable expansion parameter such that (14) is the leading approximation to the exact value of the free-energy density. Moreover, our result (14) corresponds to a new CFT in three-dimensions. Indeed, on general grounds [9, 10] one expects that the free-energy density of a CFT placed in a slab geometry with one finite dimension of length L behaves as $f_L - f_\infty = -\tilde{c}\,\Gamma(d/2)\zeta(d)/\pi^{d/2}L^d$. In $d = 2$ the parameter $\tilde{c}$ is proportional to the central charge and the conformal anomaly [11]. However, corresponding results in $d > 2$ are still unknown. In $d = 3$ one easily obtains $\tilde{c} = 3N$ for the case of N free massless four-component Dirac fermions [3]. The value of $\tilde{c}$ for the Lee-Yang edge singularity which can be read-off from (14) is larger than $3N$, implying that the corresponding CFT is non-unitary. This is in accordance with the two-dimensional results [5].

In may cause some worry that we have connected the critical behavior of a unitary theory (Gross-Neveu) with a non-unitary one. Nevertheless, this is not a direct connection. Staying within the Gross-Neveu model and starting e.g. from the low temperature *broken* phase, we do not expect the appearance of the Lee-Yang critical behavior studied above. Namely, as we raise the temperature we simply expect that the Z_2 symmetry is restored at the critical temperature $T_c = M/2\ln 2$ and then the system continues to be in the high-temperature symmetric temperature phase for all $T > T_c$. However, if we consider the Gross-Neveu model as a component of some enlarged theory, it is quite conceivable that the presence of other fields (e.g. gauge fields), or chemical potentials might account for a possible Lee-Yang critical behavior at $T > T_c$ as they could induce imaginary values for the minimum of the effective potential σ_0 [13]. Clearly, the enlarged system should still be described by a unitary theory. From this point of view, we expect our approach and results to be most suitable for discussing effects such as the recently studied symmetry nonrestoration [14], since the latter is related to an imaginary chemical potential. Our leading-N calculations reproduce the well-known mean field theory results for the Lee-Yang edge singularity critical exponents. It would then be interesting to extend our results to next-to-leading order in $1/N$ for comparison with existing numerical calculations [15].

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